

Notes on Neuron Dynamics: The Action Potential

Neural Currents I: Analysis Summary

September 5, 2003

Equation	Quantity	Prerequisite	
a. $J_m = C_m \frac{\partial V}{\partial t} + \sum_{i=K,Na,L} g_i(V,t)[V - E_i]$	Hodgkin-Huxley Eqn: Neural currents I Voltage-dependent current gates i	$I = gV$ $g = g_1 + g_2$ $I = CdV/dt$ E_i $g(V, t)$	Ohm's Law Parallel conductances Capacitance Resting potentials (c) Gates: H-H model (f)
b. $J_m = \frac{a}{2R_i} \frac{\partial^2 V}{\partial x^2}$	Cable Equation	$g(V, t)$	Gates: H-H model (f)
c. $E_i = - (kT/ze) \ln([X_i]_{out}/[X_i]_{in})$	Nernst Equation Resting potentials E_i	Nernst-Planck Equation (d)	
d. $J = - \left(\mu z [X] \frac{\partial V}{\partial x} + \frac{\mu kT}{q} \frac{\partial [X]}{\partial x} \right)$	Nernst-Planck Equation Currents from drift, diffusion	J_{drift}, J_{diff} equations	
f. $g_{Na}(t) = \bar{g}_{Na} m(t)^3 h(t)$ $g_K(t) = \bar{g}_K n(t)^4$	Gate conductance as function of gating particle states	Hodgkin-Huxley gate model	
g. $\frac{dn(t)}{dt} = \alpha_n [1 - n(t)] - \beta_n n(t)$ sim. for $m(t), h(t)$	Time-dependence of gates	Hodgkin-Huxley gate model: reaction kinetics	
h. $\beta_n(V) = \beta_n^0 e^{-(1-\gamma)zeV/kT}$ $\alpha_n(V) = \alpha_n^0 e^{\gamma zeV/kT}$	Voltage-dependence of gates	Energy barrier model Boltzmann distribution	

Key:

q charge (coulombs)

I current (charge/sec= amperes)

J current density (charge/sec-cm²)

E electric field (force/charge)

V voltage = potential energy/charge (joules/coulomb)

R resistance (to current flow; ohms Ω)

g conductance = $1/R$

σ conductivity (inherent medium conductance)

C capacitance

$$E = dV/dt$$

$$V = IR$$

$$J = \sigma E$$

$$g_{1+2 \text{ in parallel}} = g_1 + g_2$$

$$I = C dV/dt$$

Introduction to Cognitive Science for Mathematical Scientists

Cognitive Science 050.313/613

Neural Currents II: Fundamentals of Electric Currents

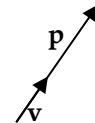
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(1) Force, energy

- a. Kinetic Energy $K \equiv \frac{1}{2}mv^2$ — [kg-m²/s² ≡ joule] *because*:
 b. $dK/dt = mv dv/dt = v ma = vF$ — by Newton's Second Law of Motion ...
 c. $dK/dt = F dx/dt$
 d. $\Delta K = F \Delta x$
 e. Potential energy (work) $U \equiv -F \Delta x$ [actually, $U \equiv -\int \mathbf{F} \cdot d\mathbf{x}$, '·' = vector inner product]
 f. Total energy $\mathcal{E} \equiv K + U$ — because:
 g. $\Delta \mathcal{E} = 0$ — Conservation of Total Energy
 h. At temperature T (in °K, degrees Kelvin; $k \equiv$ Boltzmann's constant):
 Scale of molecular energy = kT — $k = 1.38 \times 10^{-23}$ joule/°K

(2) Momentum

- a. $F = ma = m dv/dt = d(mv)/dt = dp/dt$ — [kg-m/s² ≡ newton] where
 b. $p \equiv mv$ is momentum
 c. Vector quantity: $\mathbf{p} = m\mathbf{v}$ — i.e., $(p_1, p_2, p_3) = m(v_1, v_2, v_3) = (mv_1, mv_2, mv_3)$
 d. $d\mathbf{p}/dt = \mathbf{F}$ — i.e., $(dp_1/dt, \dots) = (F_1, \dots)$
 e. $\Delta \mathbf{p} = \mathbf{F} \Delta t$ 'impulse'
 f. Conserved: $d\mathbf{p}_{\text{universe}}/dt = 0$ — why?
 g. $\mathbf{F}_{\text{on 1 from 2}} = -\mathbf{F}_{\text{on 2 from 1}}$ — Newton's First Law of Motion ...
 h. $d(\mathbf{p}_1 + \mathbf{p}_2)/dt = \mathbf{F}_{\text{on 1 from 2}} + \mathbf{F}_{\text{on 2 from 1}} = 0$ — from d



(3) Electric charge, current, force, fields, potential

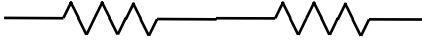

- a. Electric charge, q (for electron, e)
 b. Coulomb's law: $F = \frac{kq_1q_2}{r^2}$ — [$k = 1$ iff q in coulombs]
 c. In general: $\mathbf{F} = q\mathbf{E}$ [E in volts/m]
 d. Potential Energy difference $\equiv \Delta U$
 $= F \Delta x = qE \Delta x$
 e. Potential Energy difference *per unit charge*
 $\equiv \text{voltage drop} \equiv V \equiv \Delta U/q = E \Delta x$ [V in joules/coulomb ≡ volts]
 f. Current charge/sec: I or \mathbf{I} [coulombs/s ≡ amperes]
 g. Conservation of charge: $\sum_i I_i = 0$ — Kirchoff's Law ($\{I_i\}$ = the currents *into* a point)
 h. Current density, current/cm²: J or \mathbf{J}

(4) Resistance (R), conductance (g)

- a. $V = IR$ $I = gV$ ($g = 1/R$) — [R in ohms, Ω] Ohm's Law. "Resistor". Why?
 b. Current density J ; $I \equiv JA$
 c. Mysterious fact: $J = \sigma E$ — $\sigma =$ conductivity
 d. $V = \Delta x E = \Delta x J/\sigma = \Delta x I/A\sigma \equiv IR$; $R \equiv \Delta x/A\sigma$
 e. Why the mysterious fact? — model of electrical conductance
 i. key: scattering of e^- off lattice

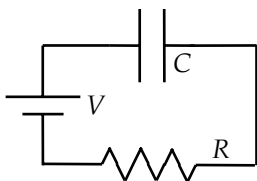
- ii. e^- collides with molecule on average after time τ (giving momentum to the lattice)
- iii. hard sphere collision: outgoing direction of e^- is random
- iv. let \mathbf{v} be the velocity of the e^- coming out of last collision, time t ago
- v. during time t gains momentum $\mathbf{F}t = e\mathbf{E}t$
- vi. momentum at end of time t is $\mathbf{p} = m\mathbf{v} + e\mathbf{E}t$
- vii. average momentum of all electrons is $\langle \mathbf{p} \rangle = m\langle \mathbf{v} \rangle + e\mathbf{E}\langle t \rangle = \mathbf{0} + e\mathbf{E}\tau$
- viii. this is m (effective [drift] velocity of e^- cloud) $\equiv m\bar{\mathbf{v}}$
- ix. $\bar{\mathbf{v}} = (e\tau/m)\mathbf{E}$ (cm/sec)
- x. suppose density of e^- s is n per cm^3
- xi. $\mathbf{J} \equiv$ current density \equiv (charge flowing through 1 cm^2 in 1 sec) = (charge in a volume $v \text{ cm}^3$)
 $= ne\bar{\mathbf{v}} = ne^2\tau/m \mathbf{E}$
- xii. $\mathbf{J} = \sigma\mathbf{E}$ $-\sigma \equiv ne^2\tau/m$
- xiii. Generalizes to other charge carriers and non-hard-sphere-collisions ($\tau \equiv$ randomization time)

(5) Resistors in the combinatorial strategy

- a. Wires: $R = 0, V = 0$
- b. Resistors in series
 - i. V adds, I in common, so ... 
 - ii. $R_{12} = R_1 + R_2$
 - iii. Check: (4)d says $R \equiv \Delta x/A\sigma$; $\Delta x_{12} = \Delta x_1 + \Delta x_2$; A, σ unchanged
- c. Resistors in parallel
 - i. I adds, V in common, so ... 
 - ii. $g_{12} = g_1 + g_2$
 - iii. Check: (4)d says $R \equiv \Delta x/A\sigma$; $A_{12} = A_1 + A_2$; $\Delta x, \sigma$ unchanged

(6) Capacitance

- a. Charge can accumulate on/in materials
- b. E.g.: parallel conducting plates; "capacitor"
- c. $Q = CV$ $-\text{due to relation between } Q \text{ and } E, \text{ Coulomb's law}$
- d. $I = dQ/dt = C dV/dt$ $-\text{ } I \text{ in to capacitor}$
- e. The basic RC circuit:
 - i. As current flows through a medium with a given R, C , equivalent to a parallel circuit
 - ii. $V/R = I = -C dV/dt$ $-\text{ ' - ' because here } I \text{ is out from capacitor}$
 - iii. $dV/dt = -(1/\tau)V$; $\tau \equiv RC$
 - iv. $V = V_0 e^{-t/\tau}$ $-\tau = \text{time constant of exponential decay, } V_0 \equiv V(t=0)$



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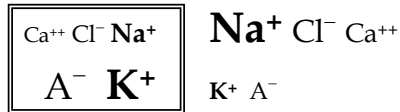
Neural Currents III: Qualitative Analysis

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(1) Background

- a. Ions most important in biological currents: K^+ , Na^+ , Cl^- , Ca^{++} ; inside cell, other anions A^-

Typical mammalian cell:



- b. Space-charge neutrality normally holds (total $+q$) = (total $-q$)

(2) Ion concentrations

- a. Are constant, even during signal propagation, to within .01%.
- b. Primary determinants:
- i. Membrane is impermeable to anions (organic acids and proteins) A^- concentrated *inside*
 - ii. Membrane is highly permeable to K^+
 - iii. Membrane is moderately permeable to Cl^-
 - iv. Axon: At resting voltages (~ -70 mV), membrane is very slightly permeable to Na^+ ; as voltage increases (above ~ -50 mV), membrane becomes increasingly permeable to Na^+ , becoming extremely permeable (high relative to K^+) at peak of action potential (variation in permeability: g_{Na} from $.05g_{\text{K}}$ to $500g_{\text{K}}$).
- c. Na-K pump
- i. 3Na^+ out for every 2K^+ in
 - ii. not a major factor in determining most resting concentrations, or short-term behavior, but ...
 - iii. essential over long term for enabling unequal concentrations to be maintained after action potentials, which involve Na^+ , K^+ currents that would eventually alter resting concentrations

(3) Consequences at rest

- a. Start with A^- , K^+ and Cl^- concentrations
- i. Space-charge neutrality outside $\Rightarrow [\text{Cl}^-]_{\text{out}} = [\text{K}^+]_{\text{out}}$
 - ii. Space-charge neutrality inside (given A^- inside) $\Rightarrow [\text{Cl}^-]_{\text{in}} < [\text{K}^+]_{\text{in}}$
- b. Can't have $[\text{K}^+]_{\text{in}} < [\text{K}^+]_{\text{out}}$
- i. For contradiction, assume so. Then have $[\text{Cl}^-]_{\text{in}} < [\text{Cl}^-]_{\text{out}} \Rightarrow \text{Cl}^-$ diffusion current inward
 - ii. therefore equilibrium $\Rightarrow \text{Cl}^-$ drift current outward \Rightarrow
 - iii. E directed inward \Rightarrow
 - iv. K^+ drift current inward; but if $[\text{K}^+]_{\text{in}} < [\text{K}^+]_{\text{out}}$ then K^+ diffusion is inward too: no equilibrium
- c. Therefore must have $[\text{K}^+]_{\text{in}} > [\text{K}^+]_{\text{out}}$ and ...
- i. must have $V \equiv V_{\text{in}} < V_{\text{out}} \equiv 0$,
 net $-$ charge on inside of *membrane*, net $+$ charge on outside: no *overall* charge in/outside and ...
 - ii. must have $[\text{Cl}^-]_{\text{out}} > [\text{Cl}^-]_{\text{in}}$.
- d. Now add Na^+ :
- i. Low membrane permeability to Na^+ means low Na^+ current, just to cancel pump Na^+ current
 - ii. Pump sends Na^+ outward (and K^+ inward, opposing its gradient)
 - iii. Diffusion Na^+ current must be inward, so ...
 - iv. must have $[\text{Na}^+]_{\text{out}} > [\text{Na}^+]_{\text{in}}$

- (4) Dendrite
- Permeability increased by neuro-transmitter-sensitive gates at synapse
 - Concentration gradients drive additional ionic current, reducing membrane charge imbalance
 - Drives V less negative/more positive at synapse;
 - Same for V nearby, since dendrite is conducting
 - V, I , decay quickly (exponentially, length constant ≈ 1 mm)
 - V s add up at beginning of axon, depolarizing it to a degree dependent on stimulation
- (5) Consequences: axon slightly depolarized ($-75 \text{ mV} \lesssim V \lesssim -50 \text{ mV}$) [indirectly by dendrites]
- Permeability of membrane to Na^+ increases slightly
 - Resting concentration gradient drives a little more Na^+ in through membrane
 - Reduces net $-$ charge on inside surface of membrane slightly
 - Drives V slightly less negative/more positive
 - Decreases slightly inward electromotive force on K^+ in membrane
 - Reduces electromotive resistance to outward K^+ diffusion
 - Increases outward K^+ flow slightly, counteracting increased inward Na^+ flow:
 - Negative feedback*
- (6) Consequences when moderately depolarized (when $V \gtrsim -50 \text{ mV}$)
- Resting concentration gradient drives Na^+ in through elevated permeability
 - Reduces net $-$ charge on inside surface of membrane
 - Drives V less negative/more positive
 - Increases permeability of membrane to Na^+ still further:
 - positive feedback*
 - Eventually inflow of Na^+ causes net inflow of total current (driven by Na^+ concentrations)
 - Na^+ permeability rapidly increases: spike
 - V repolarizes because
 - Na^+ channels spontaneously *de-activate* after being open awhile
 - Voltage-sensitive K^+ channels open (delayed relative to Na^+ channels), K^+ current cancels
- (7) Propagation
- As in dendrites, V changes at one point on axon cause V changes at adjacent points
 - large V change nearby :
 - 'down the axon': opens Na^+ gates, re-initiates spike
 - 'up the axon': Na^+ gates deactivated, K^+ gates still open, so no spike
 - Myelin
 - plugs leaky membrane, reduces capacitance: speeds passive propagation ($\tau = 1/\text{RC}$)
 - gaps – nodes of Ranvier – have gates where spike re-initiates
- (8) Gates
- Complex molecules with multiple states
 - Some states 'open' a channel, significantly increasing conductivity g
 - States described by variables ' n, m, h ' in Hodgkin-Huxley gate model – transitions between states is stochastic: probabilistic model to be discussed later in course.
 - Transition probabilities depend on voltage across membrane (unequal charge density in molecule responds to electric field)

Neural Currents IV: Equilibrium Analysis

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(1) Passive current flow: Nernst-Planck Equation

a. Given:

i. Y , an ion of valence $z \in \mathbb{Z}$ (charge $q = ze$)ii. a membrane permeable to Y , with different concentrations $[Y]_{\text{in}}, [Y]_{\text{out}}$ b. Two pressures driving flow of Y across the membrane:i. Electric field causes *drift flux* (in molecules/sec-cm², not charge/sec-cm²)

$$\mathbf{J}_{\text{drift}} = -\mu z [Y^z] \nabla V \quad - \mu = \text{mobility}$$

ii. Concentration gradient causes *diffusion flux*; Fick's Law:

$$\mathbf{J}_{\text{diff}} = -D \nabla [Y^z] \quad - D \equiv \text{diffusion coefficient}$$

$$D = (kT/e) \mu \quad - \text{Einstein (1905; Brownian motion)}$$

iii. Net result: Nernst-Planck equation

$$J = -\left(\mu z [Y] \frac{\partial V}{\partial x} + \frac{\mu kT}{e} \frac{\partial [Y]}{\partial x} \right)$$

(2) Equilibrium potential of an ion Y^z , V_m : Nernst Equation – no net Y current across membranea. $V_m \equiv V_{\text{in}} - V_{\text{out}}$, the voltage difference where $J = 0$ (J_{diff} and J_{drift} cancel – equilibrium); the *resting potential* of Y_i :

$$E_i = \frac{kt}{ze} \ln \frac{[Y_i]_{\text{out}}}{[Y_i]_{\text{in}}} \quad - \text{Nernst Equation}$$

$$\doteq (62 \text{ mV} / z) \log_{10} ([Y]_{\text{out}}/[Y]_{\text{in}}) \quad - \text{at body temperature, } 37^\circ$$

Derivation (Assuming constant $E = V_m/\Delta x$):

$$0 = J_{\text{diff}} + J_{\text{drift}} \Rightarrow -J_{\text{diff}} = J_{\text{drift}} \Rightarrow$$

$$D d[Y]/dx = -\mu z [Y] dV/dx = -\mu z [Y] E$$

$$d[Y]/dx = -c [Y] \quad c \equiv \mu z E/D = zEe/kT$$

$$[Y] = k e^{-cx}$$

$$[Y]_{\text{out}}/[Y]_{\text{in}} = e^{-c \Delta x}$$

$$\ln([Y]_{\text{out}}/[Y]_{\text{in}}) = -c \Delta x = -zE\Delta x/kT$$

$$(kT/ze) \ln([Y]_{\text{out}}/[Y]_{\text{in}}) = -E\Delta x = V_m$$

b. Since conductance is defined by $I_{\text{drift}} = gV$ and this is cancelled by I_{diff} at the resting potential E_i :

$$I_{\text{diff } i} \equiv gE_i$$

This is (approximately) constant because the ion concentrations are (approximately) constant.

c. For two permeable ions Y^n and Z^p (because there is one common V_m for all ions):

$$\left(\frac{[Y]_{\text{out}}}{[Y]_{\text{in}}} \right)^{\frac{1}{n}} = \left(\frac{[Z]_{\text{in}}}{[Z]_{\text{out}}} \right)^{\frac{1}{p}} \quad - \text{Donnan equilibrium}$$

(3) Constant Y current through membrane: Goldman-Hodgkin-Katz (GHK) model

a. Ions flow through cross-membrane protein molecules with aqueous pores

b. Assumptions (for simple pores – not complex, V -sensitive channels)

- i. Ion currents across the membrane obey the Nernst-Planck equation
 - ii. Ions do not interact with each other
 - iii. E is constant in the membrane
- c. GHK current equation:

$$I_i = P_i z_i^2 e \alpha \left(\frac{[Y_i]_{\text{in}} - [Y_i]_{\text{out}} e^{-z_i \alpha}}{1 - e^{-z_i \alpha}} \right) = P_i z_i^2 e \alpha \left(\frac{[Y]_{\text{out}} - [Y]_{\text{in}} e^{z_i \alpha}}{1 - e^{z_i \alpha}} \right)$$

where $P_i \equiv \mu_i \beta_i kT / le$, $\beta_i \equiv$ water-membrane partition coefficient for Y_i , $\alpha \equiv eV / kT$, and

Derivation: $I_i = z_i e J_i$ (I_i in coulombs/cm²; J_i in molecules/cm²; $l \equiv$ membrane thickness)

$$I_i = a_i [Y_i] - b_i d[Y_i] / dx \quad \text{Nernst-Planck: } a_i \equiv -\mu_i z_i^2 e \partial V / \partial x = \mu_i z_i^2 e V / l \quad b_i \equiv \mu_i z_i kT$$

($l > 0$ by def. when I flows out through the membrane)

$$-b_i d[Y_i] / dx = I_i - a_i [Y_i] = y \quad y \equiv I_i - a_i [Y_i] \quad \text{change variable to } y$$

$$dy / dx = dI_i / dx - a_i d[Y_i] / dx$$

$$= 0 - a_i (-1/b_i) y$$

$$= a_i (y/b_i)$$

$$y = k e^{a_i x / b_i}$$

$$a_i / b_i = (z_i e / kT) V / l = z_i \alpha / l$$

$$= k e^{z_i \alpha x / l}$$

$$k = y(0) = I - a[Y]_{x=0} \quad [Y]_{x=0} = \beta_i [Y]_{\text{in}} \quad [Y]_{x=l} = \beta_i [Y]_{\text{out}} \quad (\text{definition of } \beta_i)$$

$$I_i - a[Y]_{x=l} = y(l) = k e^{z_i \alpha} = (I_i - a_i [Y]_{x=0}) e^{z_i \alpha}$$

$$I_i - I_i e^{z_i \alpha} = a_i [Y_i]_{x=l} - a_i [Y_i]_{x=0} e^{z_i \alpha} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$I_i (1 - e^{z_i \alpha}) = a_i \beta_i [Y]_{\text{out}} - a_i \beta_i [Y]_{\text{in}} e^{z_i \alpha}$$

$$I_i = a_i \beta_i \frac{[Y_i]_{\text{out}} - [Y_i]_{\text{in}} e^{z_i \alpha}}{1 - e^{z_i \alpha}} \quad \text{where } a_i \beta_i = \mu_i z_i^2 \beta_i e V / l = P_i z_i^2 e \alpha$$

$$I_i = P_i z_i^2 e \alpha \left(\frac{[Y_i]_{\text{out}} - [Y_i]_{\text{in}} e^{z_i \alpha}}{1 - e^{z_i \alpha}} \right) = P_i z_i^2 e \alpha \left(\frac{[Y_i]_{\text{in}} - [Y_i]_{\text{out}} e^{-z_i \alpha}}{1 - e^{-z_i \alpha}} \right)$$

◇ Note: Can view as the net result of $I_{\text{out}} = \gamma [Y]_{\text{in}}$, $I_{\text{in}} = \gamma e^{-z\alpha} [Y]_{\text{out}}$, $\gamma \equiv Pz^2 e \alpha / (1 - e^{-z\alpha})$

d. Multiple ions, each $I_i =$ constant, $I_{\text{total}} = 0$ (equilibrium)

- i. $I_{\text{total}} = \sum_i I_i$, where I_i is given in (c), by the assumption that currents are independent
- ii. The resting potential of a cell with ions $Y_i \equiv \{K^+, Na^+, Cl^-\}$ and membrane permeabilities P_i

$$V_{\text{rest}} = \frac{kT}{e} \ln \frac{P_K [K^+]_{\text{out}} + P_{Na} [Na^+]_{\text{out}} + P_{Cl} [Cl^-]_{\text{in}}}{P_K [K^+]_{\text{in}} + P_{Na} [Na^+]_{\text{in}} + P_{Cl} [Cl^-]_{\text{out}}} \quad - \text{GHK voltage equation}$$

Derivation: For K^+ , Na^+ , and Cl^- , $z^2 = (\pm 1)^2 = 1$. Want $V = \alpha kT / e = (kT / e) \ln(e^\alpha)$

$$0 = I_{\text{total}} = I_K + I_{Na} + I_{Cl}$$

$$0 = I_K + I_{Na} + I_{Cl}$$

$$= P_K \gamma \frac{[K^+]_{\text{in}} - [K^+]_{\text{out}} e^{-\alpha}}{1 - e^{-\alpha}} + P_{Na} \gamma \frac{[Na^+]_{\text{in}} - [Na^+]_{\text{out}} e^{-\alpha}}{1 - e^{-\alpha}} + P_{Cl} \gamma \frac{[Cl^-]_{\text{out}} - [Cl^-]_{\text{in}} e^{-\alpha}}{1 - e^{-\alpha}}$$

$$0 = P_K ([K^+]_{\text{in}} - [K^+]_{\text{out}} e^{-\alpha}) + P_{Na} ([Na^+]_{\text{in}} - [Na^+]_{\text{out}} e^{-\alpha}) + P_{Cl} ([Cl^-]_{\text{out}} - [Cl^-]_{\text{in}} e^{-\alpha})$$

$$P_K [K^+]_{\text{in}} + P_{Na} [Na^+]_{\text{in}} + P_{Cl} [Cl^-]_{\text{out}} = (P_K [K^+]_{\text{out}} + P_{Na} [Na^+]_{\text{out}} + P_{Cl} [Cl^-]_{\text{in}}) e^{-\alpha}$$

$$V = kT / e \ln(e^\alpha) = \frac{kT}{e} \ln \left[\frac{P_K [K^+]_{\text{out}} + P_{Na} [Na^+]_{\text{out}} + P_{Cl} [Cl^-]_{\text{in}}}{P_K [K^+]_{\text{in}} + P_{Na} [Na^+]_{\text{in}} + P_{Cl} [Cl^-]_{\text{out}}} \right]$$

- ◇ Note 1: with a single ionic species, $I_{\text{total}} = I_i$, this becomes the Nernst Equation
- ◇ Note 2: one of the ion 'species' can be an ion pump \Rightarrow effect of pump on V_{rest} ($\approx 10\%$)

References:

Kandel, Eric R., Schwartz, James H., and Jessell, Thomas M. 1991. *Principles of Neural Science*. New York: Elsevier. Third Edition, Chapter 6.

Johnston, Daniel, and Wu, Samuel Miao-Sin. 1995. *Foundations of Cellular Neurophysiology*. Cambridge, MA: MIT Press.

Neural Currents V: Propagating Action Potentials

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(4) Parameters

- a = radius of cylinder; $A = \pi a^2$ = cross-sectional area of cylinder
- R_i = specific intracellular resistivity (Ω -cm)
 $R_i^{\Delta x} \equiv$ resistance to axial flow of a length Δx of cable = $R_i \Delta x / A$ ($\Omega = \Omega$ -cm \times cm/cm²)
- R_m = specific membrane resistivity (Ω -cm²)
 $R_m^{\Delta x} \equiv$ resistance to trans-membrane (radial) flow of a length Δx of cable = $R_m / 2\pi a \Delta x$ ($\Omega = \Omega$ -cm² / cm \times cm)
- C_m = specific membrane capacitance (F/cm²)
 $C_m^{\Delta x} \equiv$ capacitance of membrane of a length Δx of cable = $C_m 2\pi a \Delta x$ (F = F/cm² \times cm²)
- J_i = interior (axial) current density
 $I_i = J_i A \equiv$ interior (axial) current
- J_m = trans-membrane (radial) current density
 $I_m^{\Delta x} \equiv$ radial current along a length $\Delta x = J_m 2\pi a \Delta x$

(5) Cable equation [See Fig 4.6, (4.4.11), p. 63, Johnston and Wu]

- Ohm's Law along the cable axis:

$$\frac{\partial V}{\partial x} \Delta x = \Delta V_m = -I_i R_i^{\Delta x} = -I_i R_i \Delta x / A \Rightarrow \frac{\partial V}{\partial x} = -J_i R_i$$

- Kirchoff's law for current split between axial current and trans-membrane current

$$0 = \frac{\partial I_i}{\partial x} \Delta x + I_m = \frac{\partial J_i}{\partial x} A \Delta x + J_m 2\pi a \Delta x \Rightarrow \frac{\partial J_i}{\partial x} = -J_m (2/a)$$

- Combining:

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial J_i}{\partial x} R_i = J_m R_i (2/a) \Rightarrow J_m = \frac{a}{2R_i} \frac{\partial^2 V}{\partial x^2}$$

- Independently, if E is the equilibrium (reversal) potential for the ion in question*:

$$I_m^{\Delta x} = I_C^{\Delta x} + I_{\text{drift}}^{\Delta x} + I_{\text{diff}}^{\Delta x} = C_m^{\Delta x} \frac{\partial V_m}{\partial t} + \frac{V_m}{R_m^{\Delta x}} - \frac{E}{R_m^{\Delta x}} \Rightarrow$$

$$J_m = C_m \frac{\partial V_m}{\partial t} + \frac{V_m}{R_m} - \frac{E}{R_m} = C_m \frac{\partial V_m}{\partial t} + g_m (V_m - E) \quad g_m \equiv \frac{1}{R_m}$$

- Combining:

$$\frac{a}{2R_i} \frac{\partial^2 V}{\partial x^2} = J_m = C_m \frac{\partial V_m}{\partial t} + g_m (V_m - E)$$

- This is the *cable equation*, which can be rewritten

$$\lambda^2 \frac{\partial^2 V}{\partial x^2} = \tau_m \frac{\partial V_m}{\partial t} + (V_m - E)$$

where the characteristic length and time are respectively:

$$\lambda \equiv \sqrt{\frac{aR_m}{2R_i}} \quad \tau_m \equiv R_m C_m$$

- g. Solutions: A bear. [See Johnston & Wu, Sect. 4.4.2, pp. 66–84]
- h. *Note concerning variation of diffusion current with changing membrane conductance
 - i. As gates open/close, mobility of ions μ through them goes up/down.
 - ii. Diffusion current is proportional to D which is proportional to μ ; drift current is proportional to σ/g which is proportional to μ .
 - iii. The diffusion current at resting potential is $g_i E_i$, opposing the drift current; and as g changes with V , the diffusion current changes proportionally, remaining equal to $g_i E_i$

(6) Hodgkin–Huxley equation [See Fig 3.4, (3.4.3) p. 48; Fig 6.8, p. 149, Johnston & Wu]

- a. Cable equation with multiple ion species X_i , each with its own conductance g_i and equilibrium potential E_i
- b. In terms of conductances $g_i = 1/R_{m,i}$: (L \equiv 'leak' current \approx Cl^- current)

$$J_m = C_m \frac{\partial V}{\partial t} + \sum_{i=\text{K,Na,L}} g_i(V,t)[V - E_i]$$

(7) Equations for $g(V, t)$? Probabilistic model of ionic channels.

(8) How to solve? In general: *iterate difference equation; numerical integration*. See Fig. 6.13