050.370/670 — Formal Methods in Cognitive Science
Problem Set 3
Due 9/29/06

Problem 1

(PMW, chapter 4, exercise 2) Show that the cardinality of the set of integral powers of 10 is $\aleph_0$, i.e., $|\{10, 100, 1000, 10000, 100000, \ldots\}| = |\mathbb{N}|$.

Problem 2

Prove by induction that the following equation holds for all $n \geq 2$:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\cdots\left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Problem 3

(PMW, chapter 3, exercise 1) For each of the following relations, determine whether it is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, transitive, and connected.

i. is a child of

ii. is a brother of

iii. is a descendant of

Problem 4

Let X be the set of ordered pairs of integers (a, b) with $b \neq 0$. Show that the following relation is an equivalence relation on X:

$$(a, b) \sim (c, d) \text{ iff } a + d = b + c$$

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Problem 5

In this problem, we will consider the meanings of sentences like those in (1) with plural subjects.

(1) a. [The students] lifted the piano.
b. [The novelists] wrote books.

Assume for the purposes of this question that the meaning of such a plural subject is simply the set of entities that it describes, that is, ⟦the novelists⟧ might be equal to {Joyce, Faulkner, Calvino}. One can interpret the sentences in (1) in one of two ways. The first of these, the so-called *distributive* reading requires that each of the entities that the subject describes carry out the predicate separately. We formalize this as follows:

(2) [NP_{PL} VP] is true on a distributive reading iff for every $x \in [\![\text{NP}_{\text{PL}}]\!]$, $x \in [\![\text{VP}]\!]$.

(Recall from the last problem set that we are assuming that the meaning of a VP is the set of individuals which it describes.) If we assume that $[\![$wrote books$]\!]$ = {Joyce, Faulkner, Calvino, Buchanan, Seuss}, example (1)b is true on a distributive reading by this condition since each member of $[\![$the novelists$]\!]$ is a member of $[\![$wrote books$]\!]$. Another interpretation of the sentences in (1) involves the *collective* reading, in which the subject as a group carries out the action described by the predicate:

(3) [NP_{PL} VP] is true on a collective reading iff $[\![\text{NP}_{\text{PL}}]\!] \in [\![\text{VP}]\!]$.

Suppose now that $[\![$the students$]\!]$ = {Jones, Smith, Brown} and $[\![$lifted the piano$]\!]$ = {{Brown, Smith, Jones},{Schwarzenegger}} (i.e., Brown, Smith and Jones together lifted the piano, as did Schwarzenegger by himself). (Note that we are assuming here the VP meanings can also include sets of individuals to represent activities carried out by a group.) The condition in (3) renders example (1)a true on a collective reading, since the set $[\![$the students$]\!]$ is itself an element of $[\![$lifted the piano$]\!]$.

As is seen from these examples, different examples may favor a collective or a distributive interpretation: (1)a favors the collective reading, since individual students are not usually capable of lifting pianos on their own. In contrast, example (1)b favors the distributive reading, as novel writing is most often a solitary activity. The collective/distributive contrast becomes especially sharp in sentences like the following which are true only on one of these readings (distributive and collective, respectively), and false on the other.

(4) a. The buses in this town consume more gasoline than the cars.
b. The conventional bombs dropped in World War II did more damage than the nuclear bombs dropped in World War II.

Part A: Turn now to the following sentence:

(5) The men wrote operas.

Suppose for present purposes that the NP and VP are assigned meanings as follows:

(6) a. $[\![$the men$]\!]$ = { Mozart, Verdi, Gilbert, Sullivan }
b. $[\![$wrote operas$]\!]$ = { Mozart, Verdi, {Gilbert, Sullivan}}

This VP meaning reflects the fact that Mozart and Verdi each wrote operas on their own, while Gilbert and Sullivan only wrote operas together. Given this state of affairs, it seems a clear intuition that (5) is true. According the definitions given above in (2) and (3), is this sentence true on the collective and/or the distributive readings? State clearly why or why not.

Part B: To represent the interpretation of examples like (5) in the context provided by (6), it has been suggested that the range of interpretations for plural noun phrases needs to be expanded beyond the collective/distributive dichotomy. One possible way of doing this invokes the following condition:

(7) [NP_{PL} VP] is true iff there is a partition Π of $[\![\text{NP}_{\text{PL}}]\!]$ such that for every $\pi \in \Pi, \pi \in [\![\text{VP}]\!]$.

Recall from class that a partition is defined as follows:

(8) A is a *partition* of B iff all of the following hold:

a. $A \subseteq \wp(B)$;

b. $\emptyset \notin A$;

c. $\bigcup A = B$;

d. For all $x, y \in A$, if $x \cap y \neq \emptyset$, then $x = y$.

Assume that if a VP is true of an individual x, it is also true of the singleton $\{x\}$ (as in the meaning for ⟦lifted the piano⟧ given above). Does this definition makes (5) true given the meaning assignments in (6)? Explain why or why not.

Part C: Show how the condition in (7) also allows for strictly collective or distributive interpretations (as in (2) and (3)). To do this, provide meanings for the NPs and VPs in the examples in (1) and show what the required partitions are in each case.

Part D: Next consider the following sentence:

(9) The men wrote musicals.

Suppose that ⟦the men⟧ = {Rodgers, Hammerstein, Hart}. As it happens, Rodgers and Hammerstein wrote musicals together, as did Rodgers and Hart, but none wrote musicals on their own. This means that ⟦wrote musicals⟧ = {{Rodgers, Hammerstein},{Rodgers, Hart }}. Does this sentence come out true according to the definition in (7)? Why or why not? Can you think of a minimal change to the condition in (7) that will make (9) comes out as true? (*Hint:* Consider the following definition: A is a *cover* of B iff $A \subseteq \wp(B)$, $\emptyset \notin A$, and $\bigcup A = B$.)

Part E: Finally, consider a situation in which the meaning of the plural NP *the TAs* is {Fred, Alice, Josephine} and in which TAs are each paid \$7,000 per year. Is the following sentence judged true or false by your condition from part D?

(10) The TAs were paid exactly \$14,000 last year.

Does this accord with your intuitions?