050.370/670 — Formal Methods in Cognitive Science
Problem Set 2
Due 9/22/06

Problem 1

Let $R = \{(a,b),(a,c),(c,d),(a,a),(b,a)\}$. What is the composition of R with itself, i.e., $R \circ R$? What is R^{-1}? Are $R, R \circ R,$ and R^{-1} functions over $\{a,b,c,d\} \rightarrow \{a,b,c,d\}$ (if not, say why not)?

Problem 2

The concept of *ordered pair* that we discussed in class can be generalized to *sequences* of arbitrary length, so that $\langle 1 \rangle, \langle 9,3 \rangle, \langle 1,7,9 \rangle$, and $\langle 1,2,3,4 \rangle$ are all sequences. Given a sequence s, $|s|$ denotes the *length of* s (i.e., the number of elements in the sequence), and s_i denotes the ith member of the sequence (i.e., if $|s| = n$, then $s = \langle s_1, s_2, \ldots, s_n \rangle$). The following condition defines what it means for two sequences s, t to be identical:

$s = t$ iff $|s| = |t|$ and for all $1 \leq i \leq |s|, s_i = t_i$.

Using this definition, prove that there is only one empty sequence, (i.e., a sequence of length zero).

Problem 3

We can use sequences to formalize something that will be very useful in our investigations of mathematical approaches to language, namely (character) strings. We will formalize a string as a sequence of symbols, so that the string 'cat' is represented as the sequence ⟨c,a,t⟩. Given a basic alphabet of symbols Σ, let us define the following notation:

- Σ^n is the set of sequences of length n (for $n \in \mathbb{N}$) each of whose coordinates are drawn from Σ. That is, $s \in \Sigma^n$ iff $|s| = n$ and for all $1 \leq i \leq n, s_i \in \Sigma$.
- $\Sigma^* = \bigcup\{\Sigma^n | n \in \mathbb{N}\}$ is the set of all finite sequences over Σ.

Part A: Using this formalization, we can now define the following operation:

The *concatenation* of strings $s \in \Sigma^n$ and $t \in \Sigma^m$, which we write as $s^\frown t$, is the sequence $\langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.

Is concatenation associative? Is it commutative? In each case, if the answer is yes, prove it. If not, show a counter-example.

Part B: For a string s over Σ, we can define the linguistically useful relation 'is a prefix of' as follows:

For $s, t \in \Sigma^*$, s is a prefix of t iff there exists $u \in \Sigma^*$ such that $t = s^\frown u$.

Using this definition as a model, define the relations 'is a suffix of' and 'is an infix of'.

Problem 4

For each of the following functions, show whether it is an injection (one-to-one), a surjection (onto), and/or a bijection (both one-to-one and onto):

(a) The successor function $s : \mathbb{N} \to \mathbb{N}$ where

$$s(x) = x + 1.$$

(b) The function $I_6 : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ where

$$I_6(K) = \begin{cases} 1, & \text{if } 6 \in K; \\ 0, & \text{if } 6 \notin K. \end{cases}$$

(c) The copy function on strings over Σ, $copy : \Sigma^* \to \Sigma^*$, where

$$copy(s) = s^\frown s$$

(d) For a given set A, the complement function $C_A : \mathcal{P}(A) \to \mathcal{P}(A)$ defined as

$$C_A(K) = A - K.$$

Problem 5

Consider sentences of the following form:

(1) a. Every student lives in Baltimore.
b. No accountants like to watch Project Runway.
c. Some donkey admires Tony Blair.

For the purposes of specifying their meanings, it is useful to analyze such sentences as consisting of three components:

- a subject quantifier (Q), *every, no* and *some* in the examples above;
- a subject noun (N), *student, accountants* and *donkey* in the examples above;
- a predicate (Pred), *lives in Baltimore, like to watch Project Runway* and *admires Tony Blair* in the examples above.

Thus, sentences like those in in (1) can all be represented by the schema 'Q-N-Pred'.

Let us suppose that the meaning of an N, which we write as ⟦N⟧, is given by a set of individuals, in particular those individuals of whom the noun can be taken to be a description.

(2) ⟦*student*⟧ = {Simon, Emma, Mark, Özge, . . .}

Assume also that plural nouns are assigned the same meaning as singular nouns, so that

(3) ⟦*students*⟧ = {Simon, Emma, Mark, Özge, . . .}

Similarly, let us take a predicate's meaning, which we write as ⟦Pred⟧, to be given by the the set of individuals who are described by the predicate.

(4) ⟦*lives in Baltimore*⟧ = {Becca, Emma, Martin O'Malley, Bill Brody, . . .}

Somewhat more formally, given a set E of individuals, called the *domain of discourse*, we will take the meanings of Ns and Preds to be subsets of E, that is, ⟦N⟧, ⟦Pred⟧$\in \mathcal{P}(E)$.

It is less clear what sort of meaning should be assigned to Qs. To tease this out, let us consider the example in (1)a. Intuitively, this sentence is true if and only if for any individual x you consider (from within the domain of discourse), if x is a student, then x also lives in Baltimore. Another way to say the same thing is that the set of students must be a subset of the set of Baltimore inhabitants. Notice that the meaning of any sentence of the form '*every* N Pred' is very closely related: 'Every clarinetist squeaks' is true just in case the set of clarinetists is a subset of the set of squeakers. This suggests that the contribution of *every* to the meaning of a sentence is that of asserting a particular relationship between the meanings of N and Pred:

(5) $[\![every]\!] = \{(A, B) | A, B \subseteq E \text{ and } A \subseteq B\}$

Given this meaning assignment, we can then combine it together with N and Pred meanings using the rule in (6) to produce a meaning for the entire sentence.

(6) A sentence of the form 'Q N Pred' is true iff $([\![\text{N}]\!], [\![\text{Pred}]\!]) \in [\![\text{Q}]\!]$.

Let us assume that all quantifiers have meanings of the same sort as we proposed for *every*. This is spelled out in (7).

(7) The meaning of a Q, which we write $[\![\text{Q}]\!]$, is a relation on $\mathcal{P}(E) \times \mathcal{P}(E)$ (where E is the domain of discourse).

Your task is to specify a meaning of this form for the quantifiers *no*, *some*, *not every*, *exactly two*, *at most ten*, *at least three* and *several*. Your answers for each of these cases should look like the meaning we assigned for *every* in (5).